PROPAGATION OF INTERNAL WAVES IN AN EXPONENTIALLY STRATIFIED LIQUID ABOVE A PROTUBERANCE

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The plane linear problem of the motion of a weakly stratified ideal liquid in a polygonal region is solved in the Boussinesq approximation. The liquid is set into motion by oscillations of parts of the boundary according to a given law. Problems of this type were studied for vertical barriers in a channel in [1, 2]. In this paper we study a special type of regions of flow, i.e., regions that are invariant under tension in one direction (a channel with a protuberance, a barrier of finite thickness, etc.). A method of solving such problems in quadratures is suggested.

1. Formulation of the Problem. We consider the plane unsteady motion of an exponentially stratified liquid filling a region Ω. The liquid is initially at rest and the boundary of the region of flow consists of segments of horizontal and vertical straight lines, $\partial\Omega_H$ and $\partial\Omega_V$, with all of the vertical segments of the boundary lying on one straight line L. We choose the Cartesian x, y coordinate system so that the y axis is directed along L in the direction opposite to free fall acceleration g. By L_Ω we denote the part of L that lies in Ω; clearly, $L_\Omega = \partial\Omega_V \cup \Gamma$, where the points Γ are interior points of the region of flow. The motion of the liquid is caused by oscillations of the segments $\partial\Omega_V$ according to given law, while the segments $\partial\Omega_H$ remain motionless. The motion of the liquid must be described with the following assumptions: 1) the amplitude of the oscillations of $\partial\Omega_V$ is small in comparison with the characteristic linear dimension D of the problem: 2) the vertical dimension over which the density of the liquid varies substantially is much greater than D; 3) the Boussinesq approximation holds.

These assumptions make it possible to use a linear theory of the motion of a weakly stratified liquid. In dimensionless variables the current function $\psi(x, y, t)$ (t is the time) satisfies the relations [3]

$$\begin{gathered}\Delta\psi_{tt} + \psi_{xx} = 0 \text{ in } \Omega,\\ \psi = 0 \text{ in } \partial\Omega_H,\ \psi = f(x,\ t) \text{ on } \partial\Omega_v,\\ \psi = \psi_t = 0 \text{ for } t = -0,\\ \psi \to 0 \text{ as } x^2 + y^2 \to \infty.\end{gathered} \tag{1.1}$$

We look for the solution of the initial-value/boundary-value problem (1.1) in the class of functions describing the motion of a liquid with a finite kinetic energy. This condition includes constraints on the behavior of the solution near cusps of the boundary and at infinity. We assume that at t > 0 the function f(x, t) is absolutely integrable over t for any y.

The region Ω can be represented as the union of half-strips of finite (or infinite) depth, whose vertical boundaries lie on L. The formulated problem can be solved for each such half-strip separately, e.g., by separation of variables, if we know the value of the current function at its vertical boundary ($\psi = 0$ at the horizontal boundaries of the half-strips). The problem, therefore, will be solved essentially if $\psi(0, y, t)$ is defined on the segments Γ.

2. Method of Solution. We supplement the definition of the sought current function ψ and the known function f at t < 0 so that $\psi = 0$, f = 0 at negative times and we apply the Fourier transform of t to Eqs. (1.1). We obtain

$$\begin{gathered}(1-\omega^2)\,\psi^F_{xx} - \omega^2\psi^F_{yy} = 0 \quad \text{in } \Omega,\\ \psi^F = 0 \text{ on } \partial\Omega_H,\ \psi^F = F(y,\ \omega) \text{ on } \partial\Omega_v.\end{gathered} \tag{2.1}$$

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Here

$$\psi^F(x, y, \omega) = \int_0^\infty \psi(x, y, t)\, e^{-i\omega t} dt; \quad F(y, \omega) = \int_0^\infty f(y, t)\, e^{-i\omega t} dt.$$

Equations (2.1) must be considered separately for $|\omega| > 1$ (elliptic case) and $|\omega| < 1$ (hyperbolic case). In the first case we make the change of variables $x = (1 - \omega^{-2})^{1/2} x_1$, $y = y_1$ and the sought function $\psi^F(x_1(1 - \omega^{-2})^{1/2}, y_1, \omega) = \Psi(x_1, y_1, \omega)$. An unusual property of the indicated regions is their invariance under such types of tension.

The function Ψ is the solution ($|\omega| > 1$) of the Dirichlet problem

$$\Delta\Psi = 0 \text{ in } \Omega, \ \Psi = 0 \text{ on } \partial\Omega_H, \ \Psi = F(y_1, \omega) \text{ on } \partial\Omega_v$$

and can be constructed in quadratures by means of a conformal mapping of Ω into a canonical region. Since Ω is polygonal, this mapping can always be constructed with the aid of the Schwartz–Christoffel integral [4], if Ω is a simply connected region. The fact that Ω is not simply connected means that floating vertical plates of zero thicknesses, fastened along L, exist in the liquid. The discussion below is confined to simply connected regions of flow, but an important property, namely that the conformal mapping is independent of ω, holds in the general case as well.

Suppose that the analytic function $z = z(\zeta)$ ($z = x_1 + iy_1$, $\zeta = \xi + i\eta$) effects the conformal mapping of the upper half-plane $\eta > 0$ into the region Ω, with the same direction of travel along the boundary. Then $x_1 = x_1(\xi, \eta)$, $y_1 = y_1(\xi, \eta)$. We introduce new functions

$$W(\xi, \eta, \omega) = \Psi[x_1(\xi, \eta), y_1(\xi, \eta), \omega], \ W_0(\xi, \omega) = F[y_1(\xi, 0), \omega]$$

and note that W is the solution of the problem

$$\Delta W = 0 \ (\eta > 0), \ W = W_0(\xi, \omega) \ (\eta = 0, \ \xi \in \Gamma_v), \\ W = 0 \ (\eta = 0, \ \xi \in R^1 \backslash \Gamma_v)$$

(Γ_V is the inverse image of segments of the boundary $\partial\Omega_V$ under conformal mapping). If $\partial\Omega_V$ consists of more than one segment, then Γ_V is an unconnected set. In any event [5]

$$W(\xi, \eta, \omega) = \frac{1}{\pi} \int_{\Gamma_v} W_0(\xi_1, \omega) \frac{\eta\, d\xi_1}{(\xi - \xi_1)^2 + \eta^2}.$$

The inverse mapping $\zeta = \zeta(z)$, $\xi = \xi(x_1, y_1)$, $\eta = \eta(x_1, y_1)$ enables us to rewrite the last formula in the function Ψ:

$$\Psi(x_1, y_1, \omega) = \frac{1}{\pi} \int_{\partial\Omega_v} F(y_2, \omega) \frac{\eta(x_1, y_1)\, \xi_{y_1}(0, y_2)\, dy_2}{[\xi(x_1, y_1) - \xi(0, y_2)]^2 + \eta^2(x_1, y_1)}.$$

But it is sufficient for us to know Ψ for $x_1 = 0$, $y_1 \in \Gamma$, which corresponds to segments of Γ in the initial-value problem. Introducing the notation $\lambda(y) = \xi(0, y)$, $\nu(y) = \eta(0, y)$, we obtain

$$\Psi(0, y_1, \omega) = \frac{1}{\pi} \int_{\partial\Omega_v} F(y_2, \omega) \frac{\nu(y_1)\lambda'(y_2)\, dy_2}{[\lambda(y_1) - \lambda(y_2)]^2 + \nu^2(y_1)} \quad (y_1 \in \Gamma). \tag{2.2}$$

We recall that the segments Γ and $\partial\Omega_V$ do not intersect and, therefore, the integrand can have singularities only at the boundary points of Γ.

Assume that $\Psi(0, y_1, \omega)$ is known for all values of the parameter ω, then

$$\psi(0, y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(0, y, \omega)\, e^{i\omega t} d\omega.$$

We assume that $\psi(0, y, t) \equiv 0$ for $t < 0$, which is possible if $\Psi(0, y, \omega)$ can be continued analytically in the region $\mathrm{Im}\,\omega < 0$ and this continuation has no singular points. By virtue of (2.2) the analytic continuation of $\Psi(0, y, \omega)$ is equivalent to the analytic continuation of $F(y, \omega)$. But this function is analytic for $\mathrm{Im}\,\omega < 0$ by construction. Formula (2.2), therefore, defines $\Psi(0, y, \omega)$ analytically for $\mathrm{Im}\,\omega = 0$. For $\mathrm{Im}\,\omega = 0$, $|\mathrm{Re}\,\omega| < 1$ the function $\Psi(0, y, \omega)$ is the limit of the analytic function (2.2) as in $\mathrm{Im}\,\omega \to -0$, the passage to the limit being continuous. Formula (2.2) thus defines $\Psi(0, y, \omega)$ for $\omega \in R^1$. The application of the inverse Fourier transformation finally solves the problem.

$$\psi(0, y, t) = \frac{1}{\pi} \int_{\partial\Omega_v} f(\alpha, t) \frac{\nu(y)\,\lambda'(\alpha)\,d\alpha}{[\lambda(y) - \lambda(\alpha)]^2 + \nu^2(y)} \quad (y \in \Gamma). \tag{2.3}$$

The value of the current function on segments Γ thus is exactly the same as when stratification is not taken into account.

3. Steady-State Regime. Formula (2.3) remains valid if $f(y, t)$ does not decrease with increasing t. In particular, for $f(y, t) = a(y)e^{i\omega_0 t}$ we find $\psi(0, y, t) = A(y)e^{i\omega_0 t}$, where $A(y) = a(y)$ for $y \in \partial\Omega_V$ and

$$A(y) = \frac{1}{\pi} \int_{\partial\Omega_v} a(\alpha) \frac{\nu(y)\lambda'(\alpha)\,d\alpha}{[\lambda(y) - \lambda(\alpha)]^2 + \nu^2(y)} \quad (y \in \Gamma). \tag{3.1}$$

The last comment gives us the answer as to how to solve the problem of generation and propagation of periodic internal waves in the presence of obstacles of a special type. Namely: if the amplitude $a(y)$ and frequency ω_0 of oscillations at the vertical solid boundaries are given, the amplitude of the oscillations on segments Γ is determined from formula (3.1). After this either the Dirichlet problem for the elliptic equation (if $|\omega_0| > 1$) or the boundary-value problem for the hyperbolic equation ($|\omega_0| < 1$) is solved in the half-strips whose union gives the region of flow Ω. They are solved by the method of integral transforms or by the method of change of variables. We note that the solution of stationary problems were obtained as limits of the solutions of nonstationary initial-value/boundary-value problems for long times.

A particular case of regions of the indicated class are simply connected regions that are symmetric about the straight line L. For example, Ω is a rectilinear strip with a vertical barrier at the bottom. In relation to the proposed method this case is degenerate since for symmetric regions the linear problem under consideration can be modified and reduced to problems with mixed boundary conditions in each half-strip [1].

4. Examples. As an example let us define the function $A(y)$ in accordance with (3.1) in the problem of periodic internal waves above a protuberance. The region of flow Ω is the entire (x, y) plane without the fourth quadrant, i.e., $\partial\Omega_H = \{y = 0,\ x > 0\}$, $\partial\Omega_V = \{x = 0,\ y < 0\}$, $\Gamma = \{x = 0,\ y > 0\}$, $L = L_\Omega = \{x = 0,\ -\infty < y < +\infty\}$. The conformal mapping $\zeta(z)$ of the region of flow onto the upper half-plane has the simple form $\zeta(z) = z^{2/3}$, $0 < \arg z < 3\pi/2$. Accordingly, $\lambda(y) = y^{2/3}/2$ $(y > 0)$ and $\lambda(y) = -y^{2/3}$ $(y < 0)$, $\nu(y) = \sqrt{3}y^{2/3}/2$ $(y > 0)$. Suppose that $a(y) = \delta(y + 1)$, $\delta(y)$ is a Dirac delta function (the distance from the source of perturbation to the edge of the protuberance is taken to be the characteristic linear dimension D). Formula (3.1) gives the amplitude of the periodic oscillations on Γ:

$$A_1(y) = \frac{1}{\sqrt{3}\pi} \frac{y^{2/3}}{y^{4/3} + y^{2/3} + 1} \quad (y > 0).$$

We see that the current function is continuous on L and near the edge of the protuberance the velocity field has a singularity of the form $O(r^{-1/3})$, where $r = (x^2 + y^2)^{1/2}$, $r \to 0$, and is quadratically integrable as $y \to +\infty$, $x = 0$.

We consider the same problem when Ω is the (x, y) plane with a vertical barrier $\partial\Omega_V = \{x = \pm 0,\ y < 0\}$. Then $\partial\Omega_H = \varnothing$, $\Gamma = \{x = 0,\ y > 0\}$. The conformal mapping of Ω onto the upper half-plane has the form $\zeta(z) = e^{i\pi/4}z^{1/2}$, $-\pi/2 < \arg z < 3\pi/2$. In this problem, $\partial\Omega_V$ consists of two parts, corresponding to the two sides of a plate, whereby $\lambda(y) = \sqrt{-y}$ for $x = +0$, $y < 0$, and $\lambda(y) = -\sqrt{-y}$ for $x = -0$, $y < 0$, $\lambda(y) = 0$ for $y > 0$, $\nu(y) = \sqrt{y}$ for $y > 0$.

Formula (3.1) gives

$$A(y)=\frac{1}{\pi}\int_{-\infty}^{0} a^{-}(\alpha)\frac{\sqrt{y}}{2}\frac{d\alpha}{\sqrt{-\alpha}\,(y-\alpha)}+\frac{1}{\pi}\int_{0}^{-\infty} a^{+}(\alpha)\left(-\frac{\sqrt{y}}{2}\right)\frac{d\alpha}{\sqrt{-\alpha}\,(y-\alpha)}$$
$$(y>0)$$

[$a^-(y)$, $a^+(y)$ are the amplitudes of oscillations of the left and right sides of the barrier]. From this we have

$$A(y)=\frac{\sqrt{y}}{2\pi}\int_{-\infty}^{0}[a^{+}(\alpha)+a^{-}(\alpha)]\frac{d\alpha}{\sqrt{|\alpha|}\,(y-\alpha)}\qquad (y>0).$$

We see that $A(y) \equiv 0$ for $a^+(\alpha) = -a^-(\alpha)$, i.e., when the thickness of the barrier varies during the oscillations. To be able to compare the A(y) in the two cases considered, we assume that $a^-(y) = \delta(y + 1)$, $a^+(y) = 0$, whereby

$$A_B(y)=\frac{1}{2\pi}\frac{\sqrt{y}}{y+1}.$$

The subscript B shows that $A_B(y)$ corresponds to the problem with a vertical barrier. It has the same properties as $A_1(y)$ does in the example with a protuberance. Both of these functions reaches their maximum values for y = 1, i.e., at a point at the same distance as the source from the edge of the obstacle, hence $A_1(1) = (3^{3/2}\pi)^{-1}$ and $A_B(1) = (4\pi)^{-1}$. Moreover, $0 < A_1(y) < A_B(y)$ for $y > 0$, which can be verified directly.

For a protuberance and a barrier the motion of liquid in the left part of the region of flow ($x < 0$) is described by the same boundary-value problem, with the exception of the values of the current function at $x = 0$, $y > 0$. The inequalities obtained allow us to expect that for the same intensity of the perturbation source the amplitudes of the internal waves will be larger for the barrier than for the protuberance. The existence of horizontal boundaries results in smaller amplitudes of the internal waves.

5. Diffraction of Induced Internal Waves at a Protuberance. We consider a model initial-value/boundary-value problem,

$$\Delta\psi_{tt}+\psi_{xx}=\delta'(y)a(x-x_0)H(t)\sin(\omega_0 t)\ \text{ in }\ \Omega,$$
$$\psi=0\ \text{ on }\ \partial\Omega,\ \psi=\psi_t=0\ \text{ for }\ t=-0, \tag{5.1}$$

which in the Boussinesq approximation describes the propagation of induced internal waves above a protuberance. Here Ω is the entire (x, y) plane without the fourth quadrant ($x > 0$, $y < 0$); H(t) is the Heaviside function ($H(t) = 1$ for $t \geq 0$ and $H(t) = 0$ for $t < 0$); a(x) is a smooth finite function ($a(x) \equiv 0$ for $|x| \geq c > 0$, $-x_0 > c$). Internal waves are generated by dipoles distributed over the interval $y = 0$, $x_0 - c < x < x_0 + c$ with density $a(x - x_0)$. The wave generator begins to operate at time $t = 0$ and its intensity varies periodically with frequency ω_0 ($0 < \omega_0 < 1$). We must construct the main term of the asymptotic form of the solution of (5.1) for long times.

When the entire plane ($\Omega = R^2$) is the liquid-occupied region, we denote the solution of (5.1) by $\psi^{(1)}(x, y, t)$. Clearly, for $t > 0$ we have

$$\psi^{(1)}(x,y,t)=\psi_c^{(1)}(x,y)\sin(\omega_0 t)+\psi_H^{(1)}(x,y,t), \tag{5.2}$$

where $\psi_c^{(1)}(x, y)$ describes the periodic motion of the liquid and satisfies the inhomogeneous hyperbolic equation

$$(1-\omega_0^2)\frac{\partial^2\psi_c^{(1)}}{\partial x^2}-\omega_0^2\frac{\partial^2\psi_c^{(1)}}{\partial y^2}=a(x-x_0)\,\delta'(y), \tag{5.3}$$

and $\psi_H^{(1)}(x, y, t)$ describes a nonstationary correction for the disagreement of the periodic solution [first term in (5.2)] with the initial conditions. Here $\psi_H^{(1)}(x, y, t) \to 0$ as $t \to \infty$. The solution of Eq. (5.3) is given by the D'Alembert formula and has the form

$$\psi_c^{(1)}(x, y) = -\frac{1}{2\omega_0^2}\left[a\left(x - x_0 + \frac{\sqrt{1-\omega_0^2}}{\omega_0}y\right) + a\left(x - x_0 - \frac{\sqrt{1-\omega_0^2}}{\omega_0}y\right)\right].$$

The carrier of the smooth function $\psi_c^{(1)}(x, y)$ consists of two intersecting strips

$$S^{\pm} = \left\{x, y \mid |x - x_0 \pm \frac{\sqrt{1-\omega_0^2}}{\omega_0}y| < c\right\},$$

of which only the strip S^+ intersects the boundary of the initial region Ω. This intersection occurs along the segment

$$S_\Omega^+ = \left\{x, y \mid x = 0, \frac{\omega_0}{\sqrt{1-\omega_0^2}}(x_0 - c) < y < \frac{\omega_0}{\sqrt{1-\omega_0^2}}(x_0 + c)\right\},$$

on which

$$\psi_c^{(1)}(x, y) = -\frac{1}{2\omega_0^2}a\left(\frac{\sqrt{1-\omega_0^2}}{\omega_0}y - x_0\right).$$

The solution of the initial-value problem is

$$\psi(x, y, t) = \psi_c^{(1)}(x, y)\sin\omega_0 t + \psi_B(x, y, t) + \psi_I(x, y, t), \tag{5.4}$$

where $\psi_B(x, y, t)$ and $\psi_I(x, y, t)$ ensure satisfaction of the boundary and initial conditions, respectively. The initial-value and boundary-value problems for these correction functions are written as

$$\begin{gathered}
\frac{\partial^2}{\partial t^2}\Delta\psi_B + \frac{\partial^2}{\partial x^2}\psi_B = 0 \text{ in}\Omega, \\
\psi_B = \frac{1}{2\omega_0^2}a\left(\frac{\sqrt{1-\omega_0^2}}{\omega_0}y - x_0\right)\sin(\omega_0 t) \text{ for } x = 0, y < 0, \\
\psi_B = 0 \text{ for } x > 0, \; y = 0, \; \psi_B = \frac{\partial}{\partial t}\psi_B = 0 \text{ for } t = 0, \\
\frac{\partial^2}{\partial t^2}\Delta\psi_I + \frac{\partial^2}{\partial x^2}\psi_I = 0 \text{ in}\,\Omega, \; \psi_I = 0 \text{ in } \partial\Omega, \\
\psi_I = 0, \; \frac{\partial\psi_I}{\partial t} = -\omega_0\psi_c^{(1)}(x, y) \text{ for } t = 0.
\end{gathered} \tag{5.5}$$

At each point of the liquid-occupied region $\psi_I(x, y, t) \to 0$ as $t \to \infty$ and, hence, the main term in the asymptotic form of the solution of (5.1) for long times is determined by the first two terms in (5.4). Since the initial-value/boundary-value problem for $\psi_B(x, y, t)$ has the form (1.1) and, hence, it can be solved by the method given in Secs. 2 and 3. In particular, at long times the asymptotic form of the solution of (5.5) is

$$\psi_B(x, y, t) = \psi_B^{(c)}(x, y)\sin(\omega_0 t) + o(1),$$

where $\psi_B^{(c)}(x, y)$ satisfy

$$\frac{\partial^2}{\partial x^2}\psi_B^{(c)} = \frac{\omega_0^2}{1-\omega_0^2}\frac{\partial^2}{\partial y^2}\psi_B^{(c)}, \tag{5.6}$$

whose solution is constructed separately on the left and right sides of the flow region,

$(x < 0, -\infty < y < +\infty)$ and $(x > 0, y > 0)$, respectively. The boundary conditions

$$\psi_B^{(c)} = \frac{1}{2\omega_0^2} a\left(\frac{\sqrt{1-\omega_0^2}}{\omega_0} y - x_0\right) \ (x = -0, y < 0),\ \psi_B^{(c)} = A(y)\ (x = -0, y > 0), \tag{5.7}$$

must be satisfied at the boundary of the left part of the region $(x = -0)$ and the conditions

$$\psi_B^{(c)} = A(y) \quad (x = +0, y > 0),\ \psi_B^{(c)} = 0 \quad (x > 0, y = 0)$$

must be satisfied at the boundary of the right part. Here

$$A(y) = -\frac{n_0^2 y^{2/3}}{2\sqrt{3}\pi\omega_0^2} \int_{-c}^{c} \frac{a(\sigma)\, d\sigma}{(\sigma + x_0)^{1/3} \{n_0^4 (\sigma + x_0)^{4/3} + n_0^2 (\sigma + x_0)^{2/3} y^{2/3} + y^{4/3}\}};$$

$$n_0 = \omega_0^{1/3}/(1 - \omega_0^2)^{1/6}.$$

If the distance from the generator of internal waves to the protuberance is large or if the carrier of the function $a(x)$ is small, i.e., $c/|x_0|$ is much smaller than unity, then we have the simple asymptotic formula

$$A(y) = (2\sqrt{3}\pi\omega_0^2 |x_0|)^{-1} \frac{\zeta^2}{\zeta^4 + \zeta^2 + 1} \int_{-c}^{c} a(\sigma)\, d\sigma + \ldots,\ \zeta = \left[y\sqrt{1-\omega_0^2}/\omega_0 |x_0|\right]^{1/3}.$$

This suggests that $A(y) = O(|x_0|^{-1})$ as $|x_0| \to \infty$. The solution of the boundary-value problem (5.6), (5.7) can be written in quadratures by means of integral transforms. A method for uniquely isolating the solution of this problem is indicated in [3].

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